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Moyal phase-space analysis of nonlinear optical Kerr media

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Abstract
Nonlinear optical media of Kerr type are described by a particular version of an anharmonic quantum harmonic oscillator. The dynamics of this system can be described using the Moyal equations of motion, which correspond to a quantum phase-space representation of the Heisenberg equations of motion. For the Kerr system we derive exact solutions of the Moyal equations for an irreducible set of observables. These Moyal solutions incorporate the asymptotics of the classical limit in a simple, explicit form. An unusual feature of these solutions is that they exhibit periodic singularities in the time variable. These singularities are removed by the phase-space averaging required to construct the expectation value for an arbitrary initial state. Nevertheless, for strongly number-squeezed initial states the effects of the singularity remain observable.

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1. Introduction

Phase-space methods provide one of the most important tools to investigate the relation between classical and quantum mechanics. In classical dynamics, the state of a system can be described by a probability distribution that is a function of $x \equiv (q, p)$, i.e., of position q and momentum p of a particle. In quantum physics such a distribution cannot exist because Heisenberg's uncertainty relation prohibits simultaneous knowledge of q and p, but a number of quasi-distributions have been proposed for a phase-space analysis of quantum systems [1]. Among the most popular is the Wigner function which is related to the density matrix $\rho(q, q') = \langle q|\hat{\rho}|q'\rangle$ of a particle by

$$W(x) = \frac{1}{\pi\xi}\int_{-\infty}^{\infty} \rho(q - q', q + q')\, \mathrm{e}^{2\mathrm{i}pq'/\xi}\, \mathrm{d}q'. \tag{1}$$

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Usually the parameter ξ is replaced by $\hbar$, but for reasons explained below we use ξ instead. The Wigner function is not a probability distribution because it can have negative values, which are often interpreted as an indication for genuine quantum effects.

The Wigner function and its time evolution for a given Hamiltonian have been applied to analyze a huge variety of phenomena; in particular, during the last two decades it has been used to analyze the quantum state of light [2]. However, to investigate the classical-quantum correspondence the closely related concept of the Weyl symbol $[\widehat{A}]_{\mathrm{w}}$ of an operator $\hat{A}$ is more suitable. The Weyl symbol is an extension of the Wigner function (1), which arises when $\widehat{\rho}$ replaced by a general operator $\widehat{A}$. For the special case of the density matrix it leads to the relation $[\hat{\rho}]_{\mathrm{w}} = 2\pi\xi\, W(x)$.

For an observable, the time evolution of its Weyl symbol is governed by the Moyal equation

$$\frac{\mathrm{d}}{\mathrm{d}t} A_{\mathrm{w}}(t) = \{A_{\mathrm{w}}(t), H_{\mathrm{w}}\}_M. \tag{2}$$

Above $H_{\mathrm{w}}, A_{\mathrm{w}}(t)$ are the Weyl symbols of the Hamiltonian and $\widehat{A}(t)$, respectively. The Moyal bracket $\{\cdot,\cdot\}_M$, cf (A.4), is the quantum extension of the classical Poisson bracket. The Moyal equation corresponds to a phase-space formulation of the operator-valued Heisenberg equations of motion. The similarity between the Heisenberg equation for observables and the classical equations of motion makes the Weyl symbol representation a powerful tool to shed light on the relation between classical and quantum dynamics. The Moyal equation of motion enables one to study the dynamics of observables without reference to the quantum state of the system.

Despite these advantages the Moyal equation has not been used extensively to analyze quantum systems because it is more difficult to solve than the Schrödinger equation. For this reason not many exact solutions are known and a comparison with experimental data is often not possible. The purpose of this paper is to improve this situation by providing an exact solution of the Moyal equation for an experimentally relevant system: the Kerr model of nonlinear optics for a single-mode of the quantized radiation field. We obtain analytical solutions of the Moyal equations which are then used to characterize the transition from classical to quantum dynamics and to provide a phase-space based physical interpretation of the Kerr effect. Surprisingly it will turn out that the Weyl symbol representations for the Heisenberg picture flow of the photon creation and annihilation operators diverge periodically. We provide a systematic study of expectation values that clarifies the physical interpretation and observational consequences of these time-dependent Moyal solution singularities.

This paper is organized as follows. In section 2, we review the basic properties of the Kerr effect. Within the framework of the quantized Kerr model, the Moyal equation of motion (2) is solved exactly in section 3 for the evolution of a family of observables constructed from the creation and annihilation operators. The phase-space based classical limit is studied in section 4. The exact Weyl symbol solutions of section 3 exhibit time-periodic singularities. In section 5, we demonstrate that these singularities generate observable, *finite* peaks for certain specific squeezed coherent states of light. Furthermore, we show in section 6 that all coherent state expectation values of the Moyal equation solutions are free of singularities.

2. The Kerr model

The Kerr model of optical nonlinearities is one of the most studied systems in quantum optics. In a Kerr medium, the refractive index of a classical beam of light depends on the light intensity as $n = n^{(0)} + n^{(2)} I$, where n denotes the total refractive index, $n^{(0)}$ the linear refractive index,

I the light intensity and $n^{(2)}$ the optical Kerr coefficient. The Kerr effect is invaluable for spectral broadening and self-focusing of laser pulses [3]. It usually appears in special crystals and its magnitude is typically so small that large light intensities are needed. However, recent research on electromagnetically induced transparency [4–6] has made very large Kerr coefficients with values of up to $n^{(2)} \approx 0.1\,\mathrm{cm}^2\,\mathrm{W}^{-1}$ possible [7–10] and may even lead to nonlinear effects at the single-photon level [11–15].

A quantum description of the Kerr effect can be accomplished by replacing the intensity of light in the refractive index by the corresponding operator. This is equivalent to introducing a quartic interaction term in the radiation Hamiltonian [1]. A particularly simple description can be achieved if the photon dynamics is confined by an optical cavity with high finesse mirrors. Such cavities may support only a single light mode in a given spectral range, so that the dynamics can be described by operators $\hat{a}, \hat{a}^\dagger$ that annihilate or create a photon in the cavity mode, respectively. The photon number operator is given by $\widehat{N} = \hat{a}^\dagger\hat{a}$, and the Kerr Hamiltonian is given by the Wick-ordered operator

$$\widehat{H} = \omega_2(\hat{a}^\dagger)^2\hat{a}^2 + \omega_1\hat{a}^\dagger\hat{a}. \tag{3}$$

Physically, ω_1 is related to the linear index of refraction by $\omega_1 = kv_{\mathrm{gr}}n^{(0)}$ and ω_2 to the nonlinear refractive index by $\omega_2 = kv_{\mathrm{gr}}n^{(2)}I_0$, where v_{gr} is the group velocity of light in the medium, k its wave number and $I_0 = 2\hbar kc^2/V$ the intensity of a single photon in a cavity of volume V. We remark that this model assumes an ideal cavity that is lossless and which is small enough so that the spatial propagation of photons is well described by a stationary light mode. For a more complete description the quantum noise due to imperfect mirrors and propagation effects have to be taken into account [16].

The operators $\widehat{H}, \hat{a}$ act on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$ and satisfy harmonic oscillator commutation relations

$$[\hat{a}, \hat{a}^\dagger] = \xi I, \qquad [\hat{N}, \hat{a}] = -\xi\hat{a}, \qquad [\hat{N}, \hat{a}^\dagger] = \xi\hat{a}^\dagger. \tag{4}$$

With this notation we have introduced the real dimensionless parameter ξ that allows us to interpolate between the classical limit ($\xi = 0$) and the full quantum evolution ($\xi = 1$). The fundamental distinction between quantum and classical mechanics resides in commutivity. The product of observables in classical mechanics is Abelian whereas the product operation in quantum mechanics is noncommutative. As $\xi \to 0$, noncommuting behavior in the Kerr model is suppressed, and for $\xi = 1$ standard single-mode photon physics is recovered. Specifically, the Moyal equation of motion automatically incorporates Bohr's correspondence principle: in the limit $\xi \to 0$, the Moyal bracket becomes the Poisson bracket and equation (2) then turns into the Poisson equation of motion.

The relation between ξ and the conventional 'quantization parameter' $\hbar$ can be seen by relating the creation and annihilation operators to two Hermitian operators via $\hat{a} = (\hat{q}+\mathrm{i}\hat{p})/\sqrt{2}$ and $\hat{a}^\dagger = (\hat{q} - \mathrm{i}\hat{p})/\sqrt{2}$. For dimensionless position and momentum operators $\hat{x} \equiv (\hat{q}, \hat{p}) = (\hat{x}_1, \hat{x}_2)$, one then has

$$[\hat{x}_j, \hat{x}_k] = \mathrm{i}\xi J_{jk} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5}$$

where J is the Poisson matrix. In a coordinate parametrization where $\hat{q}$ is proportional to length and $\hat{p}$ to momentum, $[\hat{x}_j, \hat{x}_k] = \mathrm{i}\hbar J_{jk}$, so the limit $\xi \to 0$ is equivalent to letting Planck's constant $\hbar$ go to zero for a conventional quantum harmonic oscillator. We use ξ instead of $\hbar$ because in quantum optics the operators $\hat{a}, \hat{a}^\dagger$ have a different physical interpretation than for a single Schrödinger particle in a harmonic potential. As a consequence, their degree of commutativity is not controlled by $\hbar$, but rather by the mathematically introduced deformation parameter ξ.

Throughout our derivations free use is made of the Weyl symbol calculus that represents Hilbert space operators by functions in phase space. An overview of this quantum phase-space representation and its non-commutative $\star$ product is presented in appendix A.

3. The Moyal–Kerr problem and its solution

In this section we describe Heisenberg picture evolution in Weyl symbol form, identify the symmetries of the Moyal equation of motion and use these symmetries to construct an exact solution.

The Weyl symbol representation of the Hamiltonian (3) is

$$[\widehat{H}]_{\mathrm{w}} \equiv H(\xi, x) = \omega_2 \left[\tfrac{1}{4}x^4 - \xi x^2 + \tfrac{1}{2}\xi^2\right] + \omega_1 \left[\tfrac{1}{2}x^2 - \tfrac{1}{2}\xi\right], \tag{6}$$

where $x^2 = q^2 + p^2$. The ω_2 term is the phase-space form of the nonlinear interaction. The ω_1 portion is the symbol of the number operator $[\hat{N}]_{\mathrm{w}}(x) = N(x) = \frac{1}{2}(x^2 - \xi)$ and represents the evolution of non-interacting photons.

First, consider the Moyal equation for a general observable. Denote Schrödinger evolution by $U_t = \exp(-\mathrm{i}t\widehat{H}/\xi)$. Let $\widehat{\Theta}_0$ be an observable with dynamical value $\widehat{\Theta}(t) = U_t^\dagger\, \widehat{\Theta}_0\, U_t$ and Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Theta}(t) = \mathrm{i}\xi^{-1}[\widehat{H}, \widehat{\Theta}(t)]. \tag{7}$$

The Weyl symbol image of equation (7) is Moyal's equation (2). Let $\Theta(t|x) \equiv [\widehat{\Theta}(t)]_{\mathrm{w}}(x)$ be the symbol of the evolving observable, then

$$\begin{aligned}\dot{\Theta}(t|x) &= \{\Theta(t), H\}_M(x) = \mathrm{i}\xi^{-1}(H \star \Theta(t) - \Theta(t) \star H)(x)\\ &= \mathrm{i}\xi^{-1}(H(\mathcal{L}) - H(\mathcal{R}))\,\Theta(t|x).\end{aligned} \tag{8}$$

In the last identity, we employ the expression of the Moyal bracket in terms of the left and right operators $\mathcal{L}$ and $\mathcal{R}$ which are defined in equation (A.7). It converts the Moyal bracket into a differential operator acting on the target function $\Theta(t|x)$. Evaluating, $H(\mathcal{L}) - H(\mathcal{R})$ for Hamiltonian (6) one obtains the following third-order differential equation:

$$\dot{\Theta}(t|x) = -\left[\omega_2\left(x^2 - 2\xi - \frac{\xi^2}{4}\partial_x^2\right) + \omega_1\right](x \cdot J\partial_x)\Theta(t|x). \tag{9}$$

To fully characterize a quantum system, equation (9) needs to be solved for a comprehensive set of operators. For the Kerr–Moyal problem, such a set is given by $\{(\hat{a}^\dagger)^s \hat{a}^m | 0 \leqslant s, m \in \mathbb{N}\}$. This irreducible family of operators is equivalent to all polynomials in $\hat{q}$ and $\hat{p}$.

We denote by

$$\Theta_{sm}(t|x) \equiv [(\hat{a}(t)^\dagger)^s \hat{a}(t)^m]_{\mathrm{w}}(x) \tag{10}$$

the Weyl symbol of the corresponding operators in the Heisenberg picture, where $\hat{a}(t) = U_t^\dagger\, \hat{a}\, U_t$. Our task is to solve equation (9) for the set of symbols $\Theta_{sm}(t|x)$ with initial conditions $\Theta_{sm}(0|x) \equiv [(\hat{a}^\dagger(0))^s \hat{a}(0)^m]_{\mathrm{w}}(x)$.

Moyal equation (9) has the form of a Schrödinger equation over the x-variable manifold. Specifically, the function $\Theta_{sm}(t|x)$ may be considered an unnormalized 'state' over the manifold $T^*\mathbb{R} = \mathbb{R}^2$. This is a general feature of the Moyal equation and has been used to construct WKB-type asymptotic approximations [17] for $\Theta(t|x)$. The system (9) admits a standard [18] small ξ expansion because the highest order differential operator, the Laplacian ∂_x^2, is scaled by ξ^2.

Quantities like $\mathcal{L}$ and $\mathcal{R}$ act on the Weyl symbols $\Theta_{sm}(t|x)$, while operators like $\hat{a}$ act on the usual Hilbert space. To distinguish between these two cases we use a hat to denote the latter and script capital letters denote operators acting on the Hilbert space $\mathcal{H}_2 = L^2(\mathbb{R}^2, \mathrm{d}^2x)$.

It is now useful to determine the symmetries present in the equation of motion (9). In this context it is advantageous to introduce complex coordinates $z = (q + \mathrm{i}p)$ and $\partial_z = \frac{1}{2}(\partial_q - \mathrm{i}\partial_p)$. In this notation $[\hat{a}]_\mathrm{w}(x) = a(x) = z/\sqrt{2}$ and $[\hat{a}^\dagger]_\mathrm{w}(x) = a(x)^* = z^*/\sqrt{2}$. Because a is a linear function: $a \star a = a^2$, etc, giving

$$[\hat{a}^m]_\mathrm{w}(x) = 2^{-m/2} z^m, \qquad [(\hat{a}^\dagger)^s]_\mathrm{w}(x) = 2^{-s/2} z^{*s}.$$

Employing equation (A.8) it is then straightforward to show that the initial condition is

$$\begin{aligned}\Theta_{sm}(0|x) &= \overline{a}(\mathcal{L})^s a^m(x) = \left[\frac{1}{\sqrt{2}}(z^* - \xi\partial_z)\right]^s \left(\frac{z}{\sqrt{2}}\right)^m \\ &= \sum_{l=0}^{\min(s,m)} W(m,s,l)\left(-\frac{\xi}{2}\right)^l \overline{a}^{s-l}(x)\, a^{m-l}(x)\end{aligned} \tag{11}$$

with $W(m,s,l) \equiv s!m![l!(s-l)!(m-l)!]^{-1}$. For the set of operators under consideration, the Kerr–Moyal equation (9) then becomes

$$\dot{\Theta}_{sm}(t|x) = -[\omega_2\mathcal{K} + \omega_1](x \cdot J\partial_x)\Theta_{sm}(t|x) \tag{12}$$

with

$$\mathcal{K} \equiv |z|^2 - 2\xi - \xi^2\partial_z\partial_{z^*} = x^2 - 2\xi - \frac{\xi^2}{4}\partial_x^2$$

$$(x \cdot J\partial_x) = \mathrm{i}(z\partial_z - z^*\partial_{z^*}).$$

The action of the phase-space operator $(x \cdot J\partial_x)$ is similar to that of the angular momentum operator $\widehat{L}_z$ on Hilbert space. This can be easily seen in polar coordinates, $z = r\mathrm{e}^{\mathrm{i}\phi}$, where it takes the form $(x \cdot J\partial_x) = \partial_\phi$. Furthermore, the initial condition $\Theta_{sm}(0|x)$ is an eigenstate of $(x \cdot J\partial_x)$ with eigenvalue $\lambda_{sm} = \mathrm{i}(m-s)$. Because $[\mathcal{K}, (x \cdot J\partial_x)] = 0$ we can infer that $\Theta_{sm}(t|x)$ will remain an eigenstate of $(x \cdot J\partial_x)$ with the same eigenvalue. Via this eigenfunction mechanism the third-order partial differential equation (12) is reduced to second order

$$\dot{\Theta}_{sm}(t|x) = -\mathrm{i}(m-s)[\omega_2\mathcal{K} + \omega_1]\Theta_{sm}(t|x). \tag{13}$$

We remark that if $m = s$ the right-hand side of equation (13) is zero. This means that $\Theta_{mm}(t|x) = (x^2/2)^m$ is a constant of motion. This function is also a classical constant of motion because $\{H, x^{(2m)}\} = 0$.

Equation (13) has the formal solution

$$\Theta_{sm}(t|x) = \mathrm{e}^{-\mathrm{i}(m-s)\omega_1 t}\,\mathrm{e}^{-\mathrm{i}(m-s)t\omega_2\mathcal{K}}\Theta_{sm}(0|x). \tag{14}$$

We can now take advantage of the special form (11) of the initial conditions. It is well known cf [19, p 40] that $e^{\hat{K}}\hat{A} = e^r\hat{A}e^{\hat{K}}$ for $[\hat{K}, \hat{A}] = r\hat{A}$ and $r \in \mathbb{C}$. Because

$$\left[\mathcal{K}, \tfrac{1}{\sqrt{2}}(z^* - \xi\partial_z)\right] = \xi\tfrac{1}{\sqrt{2}}(z^* - \xi\partial_z)$$

we can express the formal solution as

$$\begin{aligned}\Theta_{sm}(t|x) &= 2^{-s/2}\exp(-\mathrm{i}(m-s)t(\omega_1 + \omega_2\xi s))(z^* - \xi\partial_z)^s \mathrm{e}^{-\mathrm{i}(m-s)t\omega_2\mathcal{K}}\left(\frac{z}{\sqrt{2}}\right)^m \\ &= 2^{-s/2}\exp(-\mathrm{i}(m-s)t\omega_2\xi s)(z^* - \xi\partial_z)^s\Theta_{0m}(m^{-1}(m-s)t|x).\end{aligned}$$

It is therefore sufficient to find a closed form for $\Theta_{0m}(t|x)$. To do so we make the ansatz

$$\Theta_{0m}(t|x) = \mathrm{e}^{g(t)x^2}\mathrm{e}^{-\mathrm{i}m(\omega_1 - 2\xi\omega_2)t} f(t)\, a^m(x)\ , \tag{15}$$

with initial conditions $g(0) = 0$ and $f(0) = 1$. Inserting this ansatz into equation (13) and sorting the resulting equation in powers of $|z|^2$ yields a coupled set of differential equations for $g(t)$ and $f(t)$,

$$\dot{g} = \mathrm{i}m\omega_2(-1 + \xi^2 g^2), \qquad \dot{f} = \mathrm{i}m(m+1)\xi^2\omega_2 g f,$$

which have the solutions

$$g(t) = -\frac{\mathrm{i}}{\xi}\tan(m\xi\omega_2 t), \qquad f(t) = (\sec(m\xi\omega_2 t))^{m+1}.$$

This leads to one of the main results of this work: the exact solution of the Moyal equation for the family of operators $(\hat{a}^\dagger)^s\hat{a}^m$ is given by

$$\Theta_{sm}(t|x) = \mathrm{e}^{-\mathrm{i}(m-s)\omega_1 t}\,\mathrm{e}^{\mathrm{i}(2-s)\tilde{t}}(\sec\tilde{t}\,)^{m+1}\left(\frac{z^* - \xi\partial_z}{\sqrt{2}}\right)^s \exp\left(-\frac{\mathrm{i}}{\xi}zz^*\tan\tilde{t}\right)\left(\frac{z}{\sqrt{2}}\right)^m \tag{16}$$

with $\tilde{t} \equiv (m-s)\xi\omega_2 t$. Evaluating the s-fold derivative and converting to the phase-space variables x yields

$$\Theta_{sm}(t|x) = \mathrm{e}^{-\mathrm{i}(m-s)\omega_1 t}(\sec\tilde{t}\,)^{s+m+1}\exp\left(2\mathrm{i}\,\tilde{t} - \mathrm{i}\frac{x^2}{\xi}\tan\tilde{t}\right)$$
$$\times\sum_{l=0}^{\min(s,m)} W(m,s,l)\left(-\frac{\xi}{2}\mathrm{e}^{-\mathrm{i}\tilde{t}}\cos\tilde{t}\right)^l \overline{a}^{s-l}(x)\,a^{m-l}(x). \tag{17}$$

This final form displays the adjoint symmetry: $[(\hat{a}(t)^\dagger)^s\hat{a}(t)^m]^\dagger = (\hat{a}(t)^\dagger)^m\hat{a}(t)^s$, or equivalently $\Theta_{sm}(t|x)^* = \Theta_{ms}(t|x)$.

A striking feature of the solutions (17) is that they have a singular amplitude for times whenever $\cos\tilde{t} = 0$. Henceforth we will refer to this behavior as the Moyal singularity. Its mathematical origin is that $g(t)$ obeys a nonlinear Ricatti equation.

4. Classical and quantum trajectories

The manner in which quantum phase-space solutions embed the classical dynamics occurs in two different ways. In the first way, one characterizes how the solutions $\Theta_{sm}(t|x)$ transform into the Poisson equation solutions as $\xi \to 0$. The second semiclassical association relates quantum expectation values to corresponding classical flows. The first way, the phase-space correspondence, is treated in this section.

Quantum trajectories on phase space are defined as the symbol image of the Heisenberg coordinate operator evolution: $\hat{x}(t) = U_t^\dagger\hat{x}\,U_t$, in detail

$$Z(t,\xi|x) \equiv [\hat{x}(t)]_{\mathrm{w}}(x) = ([\hat{q}(t)]_{\mathrm{w}}, [\hat{p}(t)]_{\mathrm{w}})\,(x). \tag{18}$$

The Moyal solutions above give formulae for $Z(t,\xi|x)$ via the relationships

$$[\hat{q}(t)]_{\mathrm{w}}(x) = \tfrac{1}{\sqrt{2}}(\Theta_{01}(t|x) + \Theta_{10}(t|x)) = \sqrt{2}\mathrm{Re}\Theta_{01}(t|x),$$
$$[\hat{p}(t)]_{\mathrm{w}}(x) = \tfrac{1}{\sqrt{2}}(\Theta_{01}(t|x) - \Theta_{10}(t|x)) = \sqrt{2}\mathrm{Im}\Theta_{01}(t|x).$$

The dynamics for the classical version of the Kerr problem is simple. The $\xi = 0$ part of $H(\xi,x)$ defines the classical Hamiltonian

$$H(\xi,x) = H_{\mathrm{cl}}(x) + \xi h_1(x) + \frac{\xi^2}{2!}h_2(x),$$
$$H_{\mathrm{cl}}(x) = \frac{1}{4}\omega_2 x^4 + \frac{1}{2}\omega_1 x^2 \qquad h_1(x) = -\omega_2 x^2 \qquad h_2(x) = \frac{1}{2}\omega_2.$$

The classical trajectory $Z_{\mathrm{cl}}(t|x) = (q_{\mathrm{cl}}(t|x), p_{\mathrm{cl}}(t|x))$ is then the solution of Hamilton's equation

$$\dot{Z}_{\mathrm{cl}}(t|x) = J\partial_x H_{\mathrm{cl}}(Z_{\mathrm{cl}}(t|x))$$

with initial condition $Z_{\mathrm{cl}}(0|x) = x$. The solution is $Z_{\mathrm{cl}}(t|x) = [\exp t(\omega_2 x^2 + \omega_1)J]\,x$. In matrix form this is

$$Z_{\mathrm{cl}}(t|x) = \begin{pmatrix} q_{\mathrm{cl}}(t|x) \\ p_{\mathrm{cl}}(t|x) \end{pmatrix} = \begin{pmatrix} \cos(\omega_2 x^2 + \omega_1)\,t & \sin(\omega_2 x^2 + \omega_1)\,t \\ -\sin(\omega_2 x^2 + \omega_1)\,t & \cos(\omega_2 x^2 + \omega_1)\,t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{19}$$

This is oscillatory motion with a variable frequency $\omega_2 x^2 + \omega_1$ that depends on the initial value, x. The frequency increases as the constant of motion x^2 increases.

To compare the classical and the quantum trajectory it is useful to introduce the complex quantity

$$a_{\mathrm{cl}}(t|x) \equiv \frac{1}{\sqrt{2}}(q_{\mathrm{cl}}(t|x) + \mathrm{i}p_{\mathrm{cl}}(t|x)) = \mathrm{e}^{-\mathrm{i}(\omega_2 x^2 + \omega_1)\,t}\,\frac{1}{\sqrt{2}}(q + \mathrm{i}p), \tag{20}$$

which is the classical quantity corresponding to the annihilation operator. It agrees with the predictions of the Kerr model for the complex electric field amplitude of classical light: the leading phase factor represents the phase shift that light would experience when it travels through a nonlinear medium of length $L = t v_{\mathrm{gr}}$. In an optical system, x^2 represents the mean number of photons in the cavity, which can also be expressed as the intensity of the light field in units of the intensity of a single photon in the cavity.

Employing equation (19) we can express the related quantum trajectory as

$$\Theta_{01}(t|x) = \sec^2(\xi\omega_2 t)\,\mathrm{e}^{\mathrm{i}\Phi(\xi,x,t)} a_{\mathrm{cl}}(t|x), \tag{21}$$

with the quantum phase factor

$$\Phi(\xi, x, t) \equiv 2\xi\omega_2 t + x^2(\omega_2 t - \xi^{-1}\tan(\xi\omega_2 t)). \tag{22}$$

This phase vanishes at $\xi = 0$. If one implements a power series expansion of (21) about $\xi = 0$, the result defines the semiclassical expansion of the Moyal solution. To first order in ξ one has

$$\Theta_{01}^{\mathrm{sc}}(t|x) = a_{\mathrm{cl}}(t|x)[1 + 2\mathrm{i}\omega_2 t\,\xi + O(\xi^2)]. \tag{23}$$

Expressions (21)–(23) show how the periodic quantum and classical flows are interdependent. If $|\xi\omega_2 t| \ll \pi/2$ and $|\Phi(\xi, x, t)| \ll \pi/2$ the quantum and classical trajectories nearly coincide. This limit very well describes all experiments with conventional nonlinear optical crystals for which the nonlinear refractive index $n^{(2)}$ is very small.

At the other extreme, when $\xi\omega_2 t$ is close to an odd multiple of $\pi/2$, the amplitude factor $\sec^2(\xi\omega_2 t)$ is diverging and the quantum phase rotation $\Phi(\xi, x, t)$ is undergoing near infinite oscillation. This regime should soon be experimentally accessible by using EIT-based nonlinear media [7–10]. Below we will explore the degree to which large oscillations are actually observable.

Figure 1 shows the quantum effect of the nonlinearity on the complex field amplitude $|a_{\mathrm{cl}}(t|x)|$, which is independent of x^2. It can clearly be seen that the periodic divergences disappear in the classical limit $\xi \to 0$, and that even for a fully quantized theory ($\xi = 1$) they only appear for very large nonlinear refractive indices.

Figure 2 displays the quantum phase factor Φ for a fully quantized theory ($\xi = 1$). Like the amplitude it displays a periodic divergence in time. The width of the divergences in phase space is proportional to x^2, indicating that they are an intensity-dependent effect. The divergences disappear in the classical limit $\xi \to 0$.

Generally, not everything is rapidly oscillating. Recall that x^2 is both a classical and quantum constant of motion. At the classical level this constant is recovered from the flow via $Z_{\mathrm{cl}}(t, \xi|x)^2 = x^2$. At the quantum level one has $Z(t, \xi) \star Z(t, \xi)(x) = x^2$. This latter identity can be derived from Berezin's representation (A.5) of the star product which leads to a four-dimensional Fresnel integral with value x^2.

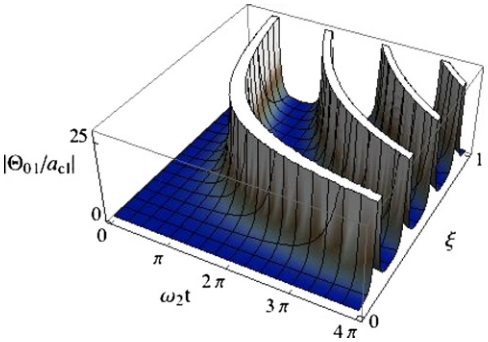

Figure 1. Magnitude of the ratio between the quantum amplitude Θ_{01} and the classical amplitude $a_{\rm cl}$.

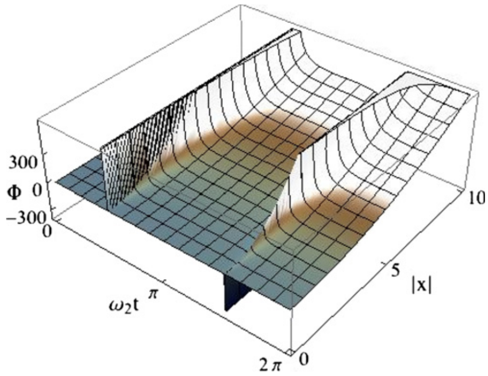

Figure 2. Quantum phase factor $\Phi(\xi, x, t)$ for $\xi = 1$.

In the literature [20, 21], the quantum trajectory $Z(t, \xi|x)$ is generally approximated by a small ξ asymptotic approximation

$$Z(t, \xi|x) = Z_{\rm cl}(t|x) + \xi z^{(1)}(t|x) + \frac{\xi^2}{2!} z^{(2)}(t|x) + \cdots. \tag{24}$$

In the Kerr–Moyal problem, and for most problems, this expansion has terms to all order in ξ. The exception occurs when H is quadratic. In this special case just the leading term $Z_{\rm cl}(t|x)$ is nonzero. For this reason little can be learned about the general nature of the classical–quantum transition by investigating quadratic Hamiltonian problems. This feature is seen in Moyal solutions above. The effect of the x-quadratic part of Hamiltonian (6) on $\Theta_{sm}(t|x)$ is confined to the phase factor $\exp(-{\rm i}(m - s)\omega_1 t)$. This factor has no ξ or x dependence.

If an exact formula for $Z(t, \xi|x)$ is known then these higher order expansion coefficients are given by

$$z^{(n)}(t|x) = \frac{\partial^n}{\partial \xi^n} Z(t, \xi|x)\Big|_{\xi=0}.$$

For example, the leading semiclassical correction to the classical Kerr problem flow is

$$z^{(1)}(t|x) = \frac{\partial}{\partial \xi} Z(t, \xi|x)\Big|_{\xi=0} = -2\,\omega_2 t\, Z_{\rm cl}(t|x). \tag{25}$$

The method for computing $z^{(n)}(t|x)$ when the full quantum trajectory is not available is to expand the Moyal equation identity in powers of ξ. This approach works if both the observable and the Weyl system Hamiltonian are semiclassical admissible, namely both admit a power series expansion about $\xi = 0$. This is the situation in the Kerr problem. The $\mathcal{O}(\xi^1)$ portion of the Moyal equation for $Z(t, \xi|x)$ is an inhomogeneous Jacobi field equation for the unknown $z^{(1)}(t|x)$, i.e.

$$\left[\frac{\rm d}{{\rm d}t} - J H''_{\rm cl}\big(Z_{\rm cl}(t|x)\big)\right] z^{(1)}(t|x) = J\partial_x h_1(Z_{\rm cl}(t|x)), \tag{26}$$

with initial condition $z^{(1)}(0|x) = 0$. The quantity H''_c is the Hessian matrix of H_c. Similar equations define the higher order corrections $z^{(n)}(t|x)$. A Jacobi field is a solution of the homogenous version of equation (26) and provides a linearized prediction for small deviations about the classical flow $Z_{\rm cl}(t|x)$.

One can readily check that $z^{(1)}(t|x) = -2\,\omega_2 t\, Z_{\rm cl}(t|x)$ is a solution to equation (26). This demonstrates the compatibility of $Z(t, \xi|x)$ in equation (18) with the standard asymptotic semiclassical expansion generated by equation (24). Formula (25) also illustrates the small time limitation of this expansion. The $z^{(1)}$ correction has unbounded growth in t; in order for the correction to be small one requires $|2\,\omega_2 t| \ll 1$. The next correction term $z^{(2)}$ grows like $x^2(\omega_2 t)^3$. This shows that the expansion (24) is non-uniform in the $(t, x) \in \mathbb{R} \times \mathbb{R}^2$ domain.

The conclusion, in the Moyal framework, that accurate semiclassical expansion of the Kerr problem is only valid for very short times agrees with Milburn's study [22] based on the Q-function representation.

5. Dynamical expectation values and their classical limit

In this section, we compute the squeezed state expectation value of the Moyal solution corresponding to $\hat{q}(t)$ and $\hat{p}(t)$ and characterize their semiclassical limits. The squeezed states are of particular interest because they will allow us to study the effects of the singularity. Furthermore, squeezed states are of high practical value because they correspond to non-classical states of light which can be used for quantum information [23]. In addition, the reduced noise of specific observables makes them of interest in high-precision experiments such as gravitational wave interferometers [24].

Squeezed states are unitary modifications of coherent states. We recall the defining equations for coherent and squeezed states. The coherent states $|\alpha\rangle$, are translated vacuum states (see, e.g., [1]). The translation operator $D(\alpha)$, $\alpha \in \mathbb{C}$, $\arg\alpha \in [0, 2\pi)$ shifts $\hat{a}$ by

$$D^\dagger(\alpha)\hat{a}D(\alpha) = \hat{a} + \alpha I, \qquad D(\alpha) \equiv \exp[\xi^{-1}(\alpha\hat{a}^\dagger - \alpha^*\hat{a})].$$

Defining $|\alpha\rangle = D(\alpha)|0\rangle$, it follows that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. The states $|\alpha\rangle$ have unit normalization with inner product $\langle\alpha|\beta\rangle = \exp\left[-\xi^{-1}\left(\frac{1}{2}|\alpha|^2 + \frac{1}{2}|\beta|^2 - \alpha^*\beta\right)\right]$.

Given the coherent states one obtains the squeezed states by the action of a unitary Bogoliubov operator, $V(\tau) \equiv \exp[\tau(\hat{a}^\dagger)^2 - \tau^*\hat{a}^2], \tau = |\tau|\exp(\mathrm{i}\phi), \phi \in [0, 2\pi)$. The squeezed states $|\tau\alpha\rangle \equiv V(\tau)|\alpha\rangle$ are the eigenfunctions of

$$V(\tau)\hat{a}V(\tau)^\dagger|\tau\alpha\rangle = \alpha|\tau\alpha\rangle.$$

Because $V(\tau)$ is unitary, the coherent state eigenvalue α and normalization $\langle\tau\alpha|\tau\beta\rangle = \langle\alpha|\beta\rangle$ are unchanged.

The coherent and squeezed states are interpreted as being near classical because they have special properties with respect to the uncertainty relations. The squeezed state mean values are readily found by exploiting the metaplectic nature of the Bogoliubov transform, specifically,

$$\begin{aligned} &V(\tau)\hat{x}V(\tau)^\dagger = S(\tau)\,\hat{x} \\ &S(\tau) = \begin{pmatrix} s\cos^2(\phi/2) + s^{-1}\sin^2(\phi/2) & -\frac{1}{2}\,(s^{-1} - s)\sin(\phi) \\ -\frac{1}{2}\,(s^{-1} - s)\sin(\phi) & s^{-1}\cos^2(\phi/2) + s\sin^2(\phi/2) \end{pmatrix}. \end{aligned} \tag{27}$$

The parameter $s \equiv \exp(-2\xi|\tau|) \leqslant 1$ describes the amount by which the uncertainty of a canonical variable can be reduced (see equation (28)). The set $\{S(\tau)|\tau \in \mathbb{C}\}$ is a family of positive symplectic matrices with inverse $S(\tau)^{-1} = S(-\tau)$ and group multiplication law $S(\tau)^2 = S(2\tau)$.

The q, p variances turn out to be [25]

$$\begin{aligned} \langle\Delta q\rangle^2_{\tau\alpha} &= \frac{\xi}{2}\left(\frac{1}{s^2}\cos^2(\phi/2) + s^2\sin^2(\phi/2)\right), \\ \langle\Delta p\rangle^2_{\tau\alpha} &= \frac{\xi}{2}\left(s^2\cos^2(\phi/2) + \frac{1}{s^2}\sin^2(\phi/2)\right). \end{aligned} \tag{28}$$

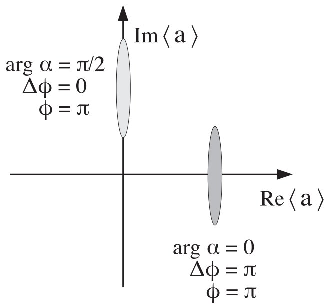

Figure 3. Representation of squeezed states in the plane of complex amplitudes. Each ellipse is centered at the mean value $\langle\hat{a}\rangle$ with the principal axes corresponding to the uncertainties of the canonical variables. The length of the axes is determined by the squeezing parameter s and the direction of the ellipse by the phase factor ϕ. The parameter $\Delta\phi$ appears in equation (35).

The uncertainty statement appropriate for this context is the Schrödinger–Robertson inequality:

$$\langle\Delta q\rangle^2\,\langle\Delta p\rangle^2 \geqslant \frac{\xi^2}{4} + \langle\widehat{F}\rangle^2, \qquad \widehat{F} \equiv \{\hat{q} - \langle\hat{q}\rangle, \hat{p} - \langle\hat{p}\rangle\}_{\mathrm{sym}}, \tag{29}$$

with the anti-commutator $\{X, Y\}_{\mathrm{sym}} \equiv XY + YX$. Employing (27) to evaluate $\widehat{F}$ gives $\langle\widehat{F}\rangle_{\tau\alpha} = (\xi/4)\,(s^{-2} - s^2)\sin\phi$.

Combining these statements shows that the $\tau\alpha$ squeezed states are minimum uncertainty states with respect to the Schrödinger–Robertson lower bound. In fact [26], a state that fulfils the equality in (29) must be a squeezed state. We remark that the phase ϕ of the squeezing parameter τ determines which of the canonical variables is squeezed: for $\phi = 0$ ($\phi = \pi$) the variance of p (q) is reduced by a factor of s, respectively. In other words, the angle variable $\phi/2$ rotates the semi-axis of the uncertainty ellipse with respect to the q, p-axis. This behavior is usually visualized by representing the squeezed state as an ellipse in the complex α-plane that indicates the uncertainties of the canonical variables, see figure 3.

The quantum phase-space representation of the expectation value is the phase-space integral (A.9)

$$\langle\hat{a}(t)\rangle_{\tau\alpha} = \langle\tau\alpha|\hat{a}(t)|\tau\alpha\rangle = \frac{1}{2\pi\xi}\int \Theta_{01}(t|x)[|\tau\alpha\rangle\langle\tau\alpha|]_{\mathrm{w}}(x)\,\mathrm{d}^2x, \tag{30}$$

where $\Theta_{01}(t|x)$ is the symbol of $\hat{a}(t) = U_t^{\dagger}\,\hat{a}\,U_t$.

Next we compute the Weyl symbol of $[|\tau\alpha\rangle\langle\tau\alpha|]_{\mathrm{w}}$ by relating it to the simpler quantity $[|\alpha\rangle\langle\alpha|]_{\mathrm{w}}$. From the coherent state wavefunction

$$\langle q|\alpha\rangle = \left(\frac{1}{\pi\xi}\right)^{1/4} \exp\left[\frac{1}{\xi}\left(-\frac{q^2}{2} + \sqrt{2}\,\alpha\,q - \alpha\,\mathrm{Re}\,\alpha\right)\right], \tag{31}$$

one obtains the associated Wigner distribution. Let $\overline{x} = (\overline{q}, \overline{p}) = (\sqrt{2}\,\mathrm{Re}\,\alpha, \sqrt{2}\,\mathrm{Im}\,\alpha)$ be the α coherent state mean values, then one has

$$[|\alpha\rangle\langle\alpha|]_{\mathrm{w}}(q, p) = 2\exp\xi^{-1}\{-(q^2 + p^2) + 2q\,\overline{q} + 2p\,\overline{p} - (\overline{q}^2 + \overline{p}^2)\}. \tag{32}$$

The Weyl symbol (32) is real because $|\alpha\rangle\langle\alpha|$ is Hermitian. The squeezed generalization of this follows from the Weyl symbol covariance property (A.10)

$$\begin{aligned}[|\tau\alpha\rangle\langle\tau\alpha|]_{\mathrm{w}}(x) &= [V(\tau)|\alpha\rangle\langle\alpha|V(\tau)^{\dagger}]_{\mathrm{w}}(x) = [|\alpha\rangle\langle\alpha|]_{\mathrm{w}}(S(\tau)\,x) \\ &= 2\exp\frac{1}{\xi}[-x\cdot S(2\tau)\,x + 2x\cdot S(\tau)\,\overline{x} - \overline{x}\cdot\overline{x}\,].\end{aligned} \tag{33}$$

The density matrix $|\tau\alpha\rangle\langle\tau\alpha|$ is projection operator that characterizes a pure ensemble of photons with mean number

$$\langle\hat{a}^\dagger\hat{a}\rangle_{\tau\alpha} = \sinh^2(2\xi|\tau|) + |\alpha\cosh(2\xi|\tau|) + \alpha^* \mathrm{e}^{\mathrm{i}\phi}\sinh(2\xi|\tau|)|^2.$$

The photon number is a constant of motion since $\hat{a}^\dagger\hat{a}$ commutes with the Kerr Hamiltonian.

The integral (30) is conveniently computed by diagonalizing $S(\tau)$ and $S(2\tau)$. The phase-space rotation

$$R(\phi) = \begin{pmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{pmatrix}, \qquad R(\phi)^T = R(\phi)^{-1} = R(-\phi)$$

achieves this via

$$S(\tau) = R(\phi)\Lambda(s)R(-\phi), \qquad \Lambda(s) = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}.$$

Note the $S(\tau)$ eigenvalues $\lambda_1 = s,\ \lambda_2 = s^{-1}$ are independent of ϕ and likewise the matrix $R(\phi)$ is independent of s. The eigenvalues are positive because $S(\tau) > 0$ and are mutual inverses since $\det S(\tau) = 1$.

Introduce the variables $y = R(-\phi)\, x$ and

$$\overline{y} = R(-\phi)\overline{x} = \begin{pmatrix} \overline{q}\cos(\phi/2) + \overline{p}\sin(\phi/2) \\ -\overline{q}\sin(\phi/2) + \overline{p}\cos(\phi/2) \end{pmatrix} = \begin{pmatrix} \overline{q}_\phi \\ \overline{p}_\phi \end{pmatrix}.$$

Changing the integration variable from x to y and employing the identity $\Theta_{01}(t|R(\phi)y) = \exp(\mathrm{i}\phi/2)\,\Theta_{01}(t|y)$ allows one to write the integral (30) as

$$\langle\hat{a}(t)\rangle_{\tau\alpha} = \frac{\mathrm{e}^{\mathrm{i}\phi/2}}{\pi\xi}\int \Theta_{01}(t|y)\, \exp\frac{1}{\xi}[-y\cdot\Lambda(s^2)\,y + 2y\cdot\Lambda(s)\,\overline{y} - \overline{y}^2]\,\mathrm{d}^2y. \tag{34}$$

In displaying the final result it is useful to use the abbreviations

$$T = \tan(\xi\omega_2 t), \qquad G(T,s) = \frac{(1+\mathrm{i}T)^2}{(1+\mathrm{i}s^{-2}T)(1+\mathrm{i}s^2T)}.$$

Integral (34) is a generalized Gaussian integral and evaluates to

$$\begin{aligned}\langle\hat{a}(t)\rangle_{\tau\alpha} = {} & \alpha\, G^{3/2}\left[\frac{1}{s}\cos\left(\xi\omega_2 t - \frac{\Delta\phi}{2}\right) + \mathrm{i}s\,\sin\left(\xi\omega_2 t - \frac{\Delta\phi}{2}\right)\right] \\ & \times \exp(-\mathrm{i}(\omega_1 t + \xi\omega_2 t - \Delta\phi/2)) \\ & \times \exp\left\{-2\mathrm{i}\frac{T}{\xi}\frac{|\alpha|^2 G}{(1+\mathrm{i}T)^2}[\mathrm{i}T + s^{-2}\cos^2(\Delta\phi/2) + s^2\sin^2(\Delta\phi/2)]\right\}.\end{aligned} \tag{35}$$

Here $\Delta\phi = \phi - 2\arg(\alpha)$ is the difference between the squeezing angle and twice the phase of the coherent state amplitude α.[4] The branch cut for $\sqrt{G}$ lies along the positive real axis.

The result above constructs the quantum mean $\langle\hat{a}(t)\rangle_{\tau\alpha}$ directly from the Moyal solution $\Theta_{01}(t|x)$. It describes in detail the dependence of the expectation value on the semiclassical scaling parameter ξ as well has the squeezing and coherent state variables, τ and α. The $\langle\hat{q}(t)\rangle_{\tau\alpha}$ and $\langle\hat{p}(t)\rangle_{\tau\alpha}$ predictions are obtained from the real and imaginary parts of $\langle\hat{a}(t)\rangle_{\tau\alpha}$. We remark that result (35) agrees with an alternative derivation that does not use phase-space techniques but employs the su(1,1) group structure of squeezing operators instead.

Formula (35) demonstrates that the functional structure of $\alpha^{-1}\langle\widehat{a(t)}\rangle_{\tau\alpha}$ with respect to four initial state parameters $\{\tau,\alpha\}$ depends on just the three variables $|\alpha|$, s, $\Delta\phi$. The exponential dependence on $|\alpha|^2$ reflects the nonlinear intensity dependence generic in the Kerr model. At $t = 0$, the case $\Delta\phi = 0$ corresponds to phase squeezing: the uncertainty of the magnitude

[4] The factor of 2 appears because a phase shift δ in the operator $\hat{a}$ changes $D(\alpha)$ to $D(\alpha\,\mathrm{e}^{-\mathrm{i}\delta})$ but $S(\tau)$ to $S(\tau\,\mathrm{e}^{-2i\delta})$.

of $\langle\hat{a}\rangle_{\tau\alpha}$ is increased and that of its phase factor is reduced. An example for phase squeezing is the lighter ellipse shown in figure 3. On the other hand, $\Delta\phi = \pi$ corresponds to number squeezing: the uncertainty of $|\langle\hat{a}\rangle_{\tau\alpha}|$, which is the square root of the mean number of photons, is decreased. This is the case for the darker ellipse shown in figure 3.

Note that solution $\Theta_{01}(t|x)$ has different frequencies for different $|x|$. Thus the minimum uncertainty character of the initial state $|\tau\alpha\rangle$ is lost for $t \neq 0$ and restored at half-period multiples, $\xi\omega_2 t = N\pi$.

An important feature of the expectation value (35) is that for fixed $s > 0$ it is a smooth bounded function in all variables; in particular, it does not display the Moyal solution singularity when $\xi\omega_2 t$ approaches $\pi/2$. In order to interpret this we recall how the Heisenberg uncertainty principle works in the quantum phase formalism. Weyl symbols are often distributions and as such do not have any restrictions on localization or magnitude. For example, the quantizer $\widehat{\Delta}(x')$ cf (A.1) is a bounded operator whose symbol is the delta function $\delta(x' - x)$. The information about phase-space uncertainty is encoded in the Wigner function $[|\tau\alpha\rangle\langle\tau\alpha|]_{\mathrm{w}}(x)$. Only when the expectation value integral (30) is evaluated are the full effects of quantum uncertainty imposed. This phase-space integration averages out the Moyal solution singularity giving a finite result.

However, one may ask whether there is a surviving signature of the Moyal singularity in the observables $\langle\hat{q}(t)\rangle_{\tau\alpha}$ or $\langle\hat{p}(t)\rangle_{\tau\alpha}$? For suitable values of τ, α the answer is yes. In the following section, we will present a general argument that these effects should be most evident for states similar to number-squeezed states, $\Delta\phi = \pi$. In figure 4, it can be seen that this is indeed the case: for strong number squeezing ($s \ll 1$, $\Delta\phi = \pi$) position and momentum mean values are significantly enhanced around the singular point.

In contrast, in the case of phase squeezing ($\Delta\phi = 0$), the singular behavior of the Moyal solution has virtually no effect on the mean amplitude (figure 5). We remark that the peak in figure 5 appears at $t = 0$ and therefore corresponds to the mean value of position and momentum in a squeezed coherent state prior to evolution.

Figure 4(*a*) suggests that a singularity does appear in the limit of infinite number squeezing. However, for any fixed value $\xi\omega_2 t \neq \pi/2$ we have $\lim_{s\to 0}\langle\hat{a}(t)\rangle_{\tau\alpha} = 0$. On the other hand, for fixed s we have

$$\lim_{\xi\omega_2 t\to\pi/2}\langle\hat{a}(t)\rangle_{\tau\alpha} = \frac{1}{s}\exp\left(-2\frac{|\alpha|^2}{\xi}\right).$$

This indicates that the peak becomes infinitely narrow in the limit of infinite squeezing. We remark that in current experiments a squeezing factor of about $s = 0.1$ can be achieved [27].

The general expectation value simplifies dramatically in several special cases. For purely coherent states, $\tau = 0$, expression (35) reduces to the well-known result (see, e.g., equation (40)) of [28])

$$\langle\hat{a}(t)\rangle_{\tau\alpha}|_{\tau=0} = \alpha\exp\left\{-\mathrm{i}\omega_1 t - 2\mathrm{i}\frac{|\alpha|^2}{\xi}\sin(\xi\omega_2 t)\,\mathrm{e}^{-\mathrm{i}(\xi\omega_2 t)}\right\}. \tag{36}$$

This no-squeezing result does not show any evidence of the Moyal solution singulary.

Next consider the small ξ behavior of $\langle\hat{a}(t)\rangle_{\tau\alpha}$. In this limit all the noncommuting effects of the Heisenberg algebra for $\hat{q}, \hat{p}$ are turned off. The Weyl symbol based semiclassical expansion is constructed by replacing Θ_{01} with its semiclassical approximation (23) in the evaluation of the integral (34). One finds

$$\langle\hat{a}(t)\rangle_{\tau\alpha} = a_{\mathrm{cl}}(t|\overline{x})\,\{1 + [2|\tau|\mathrm{e}^{\mathrm{i}\Delta\Phi} - 2|\alpha|^2\omega_2 t\,(\omega_2 t + 4\mathrm{i}|\tau|\cos(\Delta\Phi))]\xi + \mathcal{O}(\xi^2)\}. \tag{37}$$

The leading $\xi = 0$ term is just the complex statement of the Kerr classical flow (20) with the initial data $(q, p) = (\overline{q}, \overline{p}) = \sqrt{2}\,(\mathrm{Re}\,\alpha, \mathrm{Im}\,\alpha)$. Formula (37) may also be obtained by

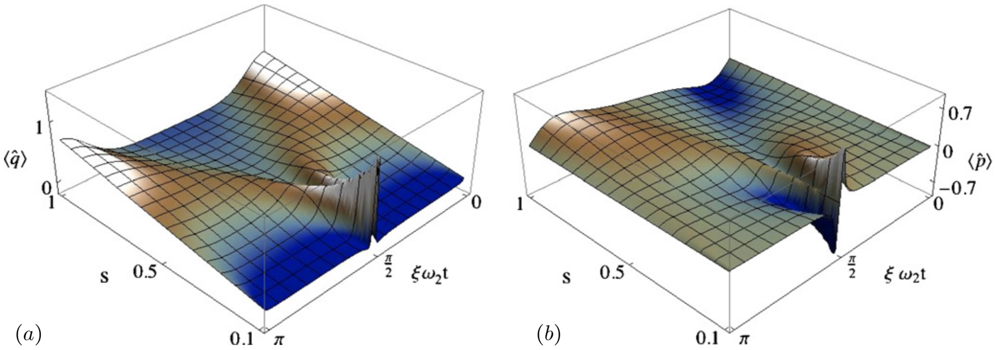

Figure 4. Mean (*a*) position and (*b*) momentum for number squeezed states ($\Delta\phi = \pi$) for the case $\alpha = \xi = 1$.

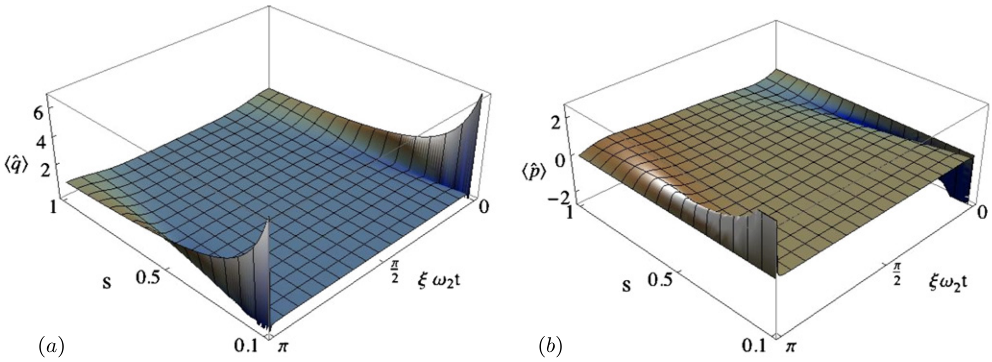

Figure 5. Mean (*a*) position and (*b*) momentum for phase squeezed states ($\Delta\phi = 0$) for the case $\alpha = \xi = 1$.

implementing a power series expansion of (35). However, the procedure using $\Theta^{\mathrm{sc}}_{01}(t|x)$ has wider application in that it does not require an exact Moyal solution. The ξ-linear correction factor senses the dependence on the squeezing $|\tau|$ and the phase $\Delta\Phi$. For this expansion to be a good approximation to $\langle\hat{a}(t)\rangle_{\tau\alpha}$ the factor 1 in the curly bracket must be much larger than the correction terms. For $|\alpha| = 1$ this requires $s \in (0.8, 1.0)$ and $|\omega_2 t| \ll 1$. Though this region of good approximation is an extremely small portion of the variable range shown in figures 4 and 5, it nevertheless covers a significant part of the experimentally accessible range.

6. Finite expectation values

The result that the Kerr dynamical flow of the Weyl symbol for both creation and annihilation operators diverge periodically in time is surprising. Such time-periodic singularities are not present in the harmonic oscillator basis or in the classical solution for the Kerr Hamiltonian. In this section, we examine for general initial states whether such singularities could in principle survive the phase-space averaging integral that defines an expectation value in the Weyl symbol picture. The results of the previous section show that the squeezed state expectation values of $\Theta_{01}(t|x)$ are finite for all times. Here we show that the expectation values of the general solution $\Theta_{sm}(t|x)$, whose amplitude diverges as $(\sec\tilde{t})^{s+m+1}$, are finite for all quantum states of interest. A quick way to show this is to use the fact that in the Heisenberg picture the dynamical annihilation operator is given by $\hat{a}(t) = \mathrm{e}^{-2\mathrm{i}\omega_2 t\hat{N}}\hat{a}(0)$, $(\omega_1 = 0)$. At the time of the singularity we have $\hat{a} = \mathrm{e}^{-\mathrm{i}\pi\hat{N}}\hat{a}(0) = \hat{P}\hat{a}(0)$, where $\hat{P}$ is the unitary parity operator [29]. Hence the condition for the expectation values to be finite is that for the state $\hat{\rho}$ the expectation values for all combinations of finite powers of $\hat{P}\hat{a}(0)$ and their adjoints are finite. This is the case for all but some exotic quantum states.

In the following, we will show how finite expectation values emerge within the Moyal representation by using the over-completeness of coherent states,

$$I = \frac{1}{\pi\xi}\int |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha.$$

We therefore can express the expectation value of an operator $\hat{f}$ as

$$\mathrm{Tr}(\rho\hat{f}) = \frac{1}{(\pi\xi)^2}\int \langle\beta|\rho|\alpha\rangle\langle\alpha|\hat{f}|\beta\rangle\,\mathrm{d}^2\alpha\,\mathrm{d}^2\beta,$$

with ρ being the density matrix of the initial state of the system. Hence, to see if the singularities can appear for any quantum state it is sufficient to investigate the matrix element $\langle\alpha|\hat{f}|\beta\rangle$ of an operator. Using equation (A.3) we can express this matrix element as

$$\langle\alpha|\hat{f}|\beta\rangle = \int f(x)\langle\alpha|\widehat{\Delta}(x)|\beta\rangle\,\mathrm{d}^2x,$$

with f being the Weyl symbol of $\hat{f}$. In the following, we will evaluate the integral over d^2x for the set of Weyl symbols $\Theta_{sm}(t|x)$. Using equations (31) and (A.2) it is not hard to see that, in complex coordinates $z = q + \mathrm{i}p$,

$$\langle\alpha|\widehat{\Delta}(x)|\beta\rangle = \frac{1}{\pi\xi}\exp\left(-\frac{zz^*}{\xi} + \frac{\sqrt{2}}{\xi}(\beta z^* + \alpha^* z) + C\right) \tag{38}$$

where $C \equiv -[|\alpha|^2 + |\beta|^2 + 2\beta\alpha^*]/(2\xi)$.

Using equations (16), (38), $\mathrm{d}^2x = \frac{1}{2}\,\mathrm{d}z\,\mathrm{d}z^*$ and (A.9) this leads to

$$\begin{aligned}\langle\alpha|(\hat{a}(t)^\dagger)^s\hat{a}(t)^m|\beta\rangle &= \int \Theta_{sm}(t|x)\langle\alpha|\widehat{\Delta}(x)|\beta\rangle\,\mathrm{d}^2x\\ &= \mathrm{e}^{-\mathrm{i}(m-s)\omega_1 t}(\sec\tilde{t})^{m+1}\,\mathrm{e}^{\mathrm{i}(2-s)\tilde{t}}\\ &\quad\times\frac{1}{2}\int\langle\alpha|\widehat{\Delta}(x)|\beta\rangle\left(\frac{z^* - \xi\partial_z}{\sqrt{2}}\right)^s\exp\left(-\frac{\mathrm{i}}{\xi}zz^*\tan(\tilde{t})\right)\left(\frac{z}{\sqrt{2}}\right)^m\mathrm{d}z\,\mathrm{d}z^*\\ &= \mathrm{e}^{-\mathrm{i}(m-s)\omega_1 t}(\sec\tilde{t})^{m+1}\,\mathrm{e}^{\mathrm{i}(2-s)\tilde{t}}\,(\alpha^*)^s\\ &\quad\times\frac{1}{2}\int\langle\alpha|\widehat{\Delta}(x)|\beta\rangle\exp\left(-\frac{\mathrm{i}}{\xi}zz^*\tan(\tilde{t})\right)\left(\frac{z}{\sqrt{2}}\right)^m\mathrm{d}z\,\mathrm{d}z^*,\end{aligned}$$

where we have performed a partial integration and used that $(z^* + \xi\partial_z)\langle\alpha|\widehat{\Delta}(x)|\beta\rangle = \sqrt{2}\alpha^*\langle\alpha|\widehat{\Delta}(x)|\beta\rangle$. Performing the integration yields

$$\begin{aligned}\langle\alpha|(\hat{a}(t)^\dagger)^s\,\hat{a}(t)^m|\beta\rangle &= \frac{(\alpha^*)^s}{2\pi\xi}(\sec\tilde{t})^{m+1}\,\mathrm{e}^{-\mathrm{i}(m-s)\omega_1 t}\,\mathrm{e}^{C+\mathrm{i}(2-s)\tilde{t}}\left(\frac{\xi}{2}\partial_{\alpha^*}\right)^m\\ &\quad\times\int\mathrm{e}^{-\frac{zz^*}{\xi}(1+\mathrm{i}\tan\tilde{t})+\frac{\sqrt{2}}{\xi}(\beta z^*+\alpha^* z)}\,\mathrm{d}z\,\mathrm{d}z^*\\ &= (\alpha^*)^s\beta^m\,\mathrm{e}^{-\mathrm{i}(m-s)(\omega_1+(m+s-1)\xi\omega_2)\,t}\\ &\quad\times\exp\left(-\frac{1}{2\xi}(|\alpha|^2+|\beta|^2) + \frac{\beta\alpha^*}{\xi}\,\mathrm{e}^{-2\mathrm{i}(m-s)\xi\omega_2 t}\right).\end{aligned} \tag{39}$$

This result is in perfect agreement with the corresponding expression derived from the solution of the ordinary Heisenberg equations of motion and thus demonstrates the consistency of the Moyal–Kerr solution (17). For our discussion it is important to note that none of these matrix elements contains a singularity. This means that the singularity can be considered as a feature of the Moyal representation of quantum mechanics that does not generate divergent expectation values.

An intuitive explanation of why the singularity does not show up in expectation values is as follows. At the time $\tilde{t}$ when the amplitude of equation (17) diverges, the phase factor diverges as well. However, the diverging factor in the phase contains the mean photon number zz^*. Consequently, the phase divergence is different for states with different photon numbers. Therefore, for $\tilde{t}$ sufficiently close to the singular value $\pi/2$, the phase factor would average to zero for any state that has a variance in the number of photons.

Hence, the only chance to see the singularity would be in a state where the number of photons is exactly, known, i.e., number states. But number states correspond to states for which the phase is completely undetermined, so that any expectation value with $s \neq m$ would average to zero for any value of $\tilde{t}$. Hence, the uncertainty relation $\Delta n \Delta\phi > 1/2$ (see, e.g., [30]) for photon number and phase prohibits the appearance of the Moyal singularity in quantum-optical experiments.

Finally, it is instructive to relate the appearance of the peak for number-squeezed coherent states to the phenomenon of quantum revivals [31–35]. In the Kerr Hamiltonian context these revivals are discussed by Tata *et al* [36] and Toscano *et al* [37]. Revivals in quantum mechanics generally appear when several transition amplitudes interfere constructively at some time during the evolution. For the Hamiltonian (3) the evolution of a pure quantum state in the number state basis $|n\rangle$ is given by

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{i}\omega_2 t n(n-1)} \psi_n |n\rangle, \tag{40}$$

where $\psi_n = \langle n|\psi(0)\rangle$ and $\omega_1 = 0$ for simplicity. At time $\omega_2 t = \pi/2$ the phase factors are all integer multiples of π in such a way that the phase factor for $n+2$ is odd if that for n is even and vice versa. So, if all the complex parameters ψ_n have the same phase, one has destructive interference between n and $n+2$ at this time. This is the case for the phase squeezed state of figure 5 where both α and τ are positive real numbers. On the other hand, if the phases of ψ_n and ψ_{n+2} differ by π, we have constructive interference at time $\omega_2 t = \pi/2$ so that a revival can appear. Hence from the Schrödinger picture point of view the peak structures in figure 4 can be interpreted as a revival phenomenon while from the Heisenberg picture, Weyl symbol point of view it represents an outcome that originates from the phase-space singularity of $\Theta_{01}(t|x)$.

In terms of Weyl symbols this equivalence between Schrödinger and Heisenberg pictures is the identity

$$\int \Theta_{sm}(t|x)\, \rho(0|x)\, \mathrm{d}^2x = \int \Theta_{sm}(0|x)\, \rho(t|x)\, \mathrm{d}^2x \tag{41}$$

where $\rho(t|x) = [|\psi(t)\rangle\langle\psi(t)|]_{\mathrm{w}}(x)$. So all the observable consequences of time-dependent revivals in the Wigner function $\rho(t|x)$ are also captured by the periodic behavior of $\Theta_{sm}(t|x)$. There is some asymmetry in this equivalence. The phase-space distribution $\rho(t|x)$ will depend on the initial state $|\psi(0)\rangle$, whereas $\Theta_{sm}(t|x)$ is state independent.

7. Conclusion

In this paper, we have derived an exact solution (16) for the Weyl symbol phase-space representation of the Kerr model of nonlinear quantum optics. For the family of observables

$\hat{a}(t)^s(\hat{a}(t)^\dagger)^m$ these solutions are given in terms of elementary functions and have periods $T = 2\pi/\omega_2(m-s)$, $m \neq s$. The natural semiclassical limit for the Moyal equation is obtained by scaling the commutation relation $[\hat{a}, \hat{a}^\dagger] = \xi I$ with a deformation parameter ξ. In the limit $\xi \to 0$ the Moyal bracket becomes the Poisson bracket and the dynamics becomes classical. Two versions of this semiclassical limit are obtained: a phase-space, symbol representation and an expectation value realization. Both of these, mutually consistent, approximations are accurate in a time regime that is very short relative to the full period. However, this restricted short-time domain is large enough to include most current experiments.

The Moyal solutions for $\Theta_{sm}(t|x)$ have periodic singularities proportional to $(\cos(m-s)\,\xi\omega_2 t)^{-s-m-1}$. These half-period singularities are absent in the classical solution. In order to obtain dynamical expectation values associated with $\Theta_{sm}(t|x)$ one must carry out a phase-space integral with respect to the static density matrix that characterizes the initial quantum state. This averaging makes expectation values free of singularities in time. However for number-squeezed states the half-period singularity leads to observable, finite, time-dependent peaks, cf figures 4(*a*) and (*b*).

A number of open questions remain. The single-mode Kerr model that we have studied gives only a good description for photons in optical cavities of extremely high finesse. A generalization of our results for a multi-mode theory of propagating photons would therefore be desirable. Alternatively, an imperfect cavity could be modeled by studying a Kerr model that is coupled to the environment and exhibits Langevin noise. Both of these aspects have been addressed by Kärtner *et al* [38] and by Stobińska *et al* [39] in the context of a specific noise model using the Wigner function. The description of multimode and decoherence phenomena in the Weyl symbol representation could provide a deeper understanding of the singularities discussed here.

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Appendix A. Weyl symbol quantum mechanics

This appendix summarizes properties of the quantum phase-space method that are employed in this paper. We collect the various known Weyl symbol identities in a notation suitable for quantum optics. The account below closely matches that found in [20, 40].

For a single-mode photon and a suitable choice of reference point, the electric field strength E is proportional to $\hat{a} + \hat{a}^\dagger$ (or equivalently to $\hat{q}$) and the magnetic field strength B to $\mathrm{i}(\hat{a} - \hat{a}^\dagger)$ or $\hat{p}$. For this reason, the phase-space based manifold for a single mode state is the real line $\mathbb{R}$. The noncommutivity of $\hat{q}$ and $\hat{p}$ arises from the mode operators $\hat{a}$ and $\hat{a}^\dagger$. The quantum state space is that spanned by the harmonic oscillator basis, or equivalently the Hilbert space of one-dimensional square integrable wavefunctions, $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$. Likewise, the associated classical phase space is $T^*\mathbb{R} = \mathbb{R}^2$ equipped with the standard Poisson bracket.

Weyl quantization maps functions on $T^*\mathbb{R}$ into operators on $\mathcal{H}$. A unified characterization of both quantization and de-quantization is achieved via a *quantizer* [41–43]. Let $\{\widehat{\Delta}(x) : x = (q, p) \in T^*\mathbb{R}\}$ be a x-dependent family of bounded, self-adjoint operators on $\mathcal{H}$ defined by their action on a wavefunction, ψ

$$\psi'(q') = [\widehat{\Delta}(q,p)\psi](q') \equiv \frac{1}{\pi\xi}\exp\left(\frac{2\mathrm{i}}{\xi}p\,(q'-q)\right)\psi(2q-q'), \qquad \text{(A.1)}$$

or, equivalently, as an integral kernel

$$\langle q'|\widehat{\Delta}(x)|q''\rangle = \frac{1}{\pi\xi}\exp\left(\frac{i}{\xi}p(q'-q'')\right)\delta(2q-q'-q'')\ . \qquad \text{(A.2)}$$

Then both quantization and de-quantization are constructed from $\widehat{\Delta}(q,p)$ via

$$\hat{f} = \int_{T^*\mathbb{R}} f(x)\,\widehat{\Delta}(x)\,\mathrm{d}^2x, \qquad [\hat{f}]_{\mathrm{w}}(x) = (2\pi\xi)\,\mathrm{Tr}\,\hat{f}\,\widehat{\Delta}(x), \qquad \text{(A.3)}$$

where Tr is the trace on $\mathcal{H}$. This pair of linear transformations are mutual inverses, so $f = [\hat{f}]_{\mathrm{w}}$. The notation $[\hat{f}]_{\mathrm{w}}$ indicates the Weyl symbol of the operator $\hat{f}$. The second identity in equation (A.3) is proportional to the Wigner transform of $\hat{f}$, cf (1).

This bijective correspondence between phase-space functions and operators is simple in a variety of important cases. For example, operators $f(\hat{q}), g(\hat{p})$ and the identity on $\mathcal{H}$ have symbols $f(q), g(p)$ and the constant function 1. The quantizer has symbol, $[\widehat{\Delta}(x)]_{\mathrm{w}}(x') = \delta(x-x')$; in turn, this implies that the exponential operator $\mathrm{e}^{\mathrm{i}u\cdot\hat{x}}$, $u \in \mathbb{R}^2$ has the symbol $\mathrm{e}^{\mathrm{i}u\cdot x}$.

The Weyl symbol framework is a Hilbert algebra, namely a complete linear space L with three basic structures: an associative product $\star$, an involution * and an inner product $(\cdot,\cdot)_L$. The product of operators on $\mathcal{H}$ is mirrored by the noncommutative product of Weyl symbols. This star product is defined by $f \star g \equiv [\hat{f}\,\hat{g}]_{\mathrm{w}}$. Given the $\star$ product, the Moyal bracket is defined as

$$\{f,g\}_M = \frac{1}{\mathrm{i}\xi}[\hat{f},\hat{g}]_{\mathrm{w}} = \frac{1}{\mathrm{i}\xi}\big(f\star g - g\star f\big). \qquad \text{(A.4)}$$

The $\star$ product has three useful representations.

The first is Berezin's integral form [44]

$$f\star g(x) = \frac{1}{(\pi\xi)^2}\iint f(x_1)\,g(x_2)\exp\left\{\frac{2\mathrm{i}}{\xi}(x_1\wedge x_2 + x_2\wedge x + x\wedge x_1)\right\}\mathrm{d}^2x_1\,\mathrm{d}^2x_2. \qquad \text{(A.5)}$$

Here $x_1\wedge x_2 \equiv x_1\cdot Jx_2$.

Next is Groenewold's derivative expansion [45]

$$f\star g(x) = \exp\left(\frac{\mathrm{i}\xi}{2}\partial_1\cdot J\partial_2\right)\prec f,g\succ(x). \qquad \text{(A.6)}$$

Above ∂_1 and ∂_2 are gradients acting on the first (f) and second (g) arguments of the product $\prec f,g\succ$. The Poisson bracket $\{f,g\}$, in this notation, is $\partial_1\cdot J\partial_2\prec f,g\succ$ followed by diagonal evaluation, $x_1 = x_2 = x$. In the case where f, g are ξ independent and suitably smooth, the series expansion of equation (A.6) gives a ξ-asymptotic expansion of $f\star g$ with the Poisson bracket term as the leading semiclassical correction.

The third realization of the $\star$ product is the left and right regular representation of the Heisenberg algebra. Define

$$\mathcal{L} \equiv x + \frac{\mathrm{i}\xi}{2}J\partial_x, \qquad \mathcal{R} \equiv x - \frac{\mathrm{i}\xi}{2}J\partial_x, \qquad \text{(A.7)}$$

then for smooth f, g

$$f\star g\,(x) = (f(\mathcal{L})\,g)\,(x) = (g(\mathcal{R})\,f)\,(x). \qquad \text{(A.8)}$$

The differential operators $\mathcal{L}$ and $\mathcal{R}$ commute.

The involution operation on L is complex conjugation. It is the symbol analog of the adjoint operation on $\mathcal{H}$. If $f = [\hat{f}]_{\mathrm{w}}$ then $f^* = [\hat{f}^\dagger]_{\mathrm{w}}$.

For suitably restricted $\hat{f}, \hat{g}$ (e.g. both Hilbert–Schmidt), the trace of the operator product defines the L inner product, in detail

$$(2\pi\xi)\,\mathrm{Tr}\,\hat{f}\,\hat{g} = \int_{\mathbb{R}^2} f \star g\,(x)\,\mathrm{d}^2x = \int_{\mathbb{R}^2} f(x)\,g(x)\,\mathrm{d}^2x = (f^*, g)_L. \tag{A.9}$$

This formula shows that quantum expectation values in L are obtained by phase-space integration. For example, this occurs if $\hat{f}$ is a density matrix and $\hat{g}$ is any observable.

Weyl quantization has a simple covariance property. A unitary operator V is called *metaplectic* if $V\hat{x}\,V^\dagger = S\hat{x}$ for some symplectic matrix S, i.e. $SJS^T = J$. If $\hat{f}$ has the symbol f, the affine canonical covariance property [20] is the statement that

$$[V\hat{f}\,V^\dagger]_{\mathrm{w}}(x) = f(Sx). \tag{A.10}$$

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